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LETTER TO THE EDITOR

The large-N limit of the threshold values in Mandelbrot's fractal percolation process†

J T Chayes and L Chayes

Department of Mathematics, University of California, Los Angeles, CA 90024, USA

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Abstract. We consider Mandelbrot's fractal percolation process, characterised by a density parameter $p \in (0, 1)$ and an integer subdivision index $N > 1$. For each N, the process is known to have a percolation transition at density $p_c(N) \in (0, 1)$. We prove that

$$\lim_{N \to \infty} p_c(N) = p_c$$

where p_c is the threshold value of the ordinary square lattice site percolation model.

Mandelbrot (1974, 1983) introduced and studied a continuum percolation problem which generates random fractal structures. In the original work, the model was called the canonical curdling process; here we will refer to it as Mandelbrot's fractal percolation process. Roughly speaking, the process is an iteration of a construction in which the unit square is divided into N^2 squares of equal area, each of which is retained independently with probability p. Mandelbrot (1983) proposed that, for any $N \geq 2$, the model should have a percolation transition at density $p_c(N) \in (0, 1)$. Chayes *et al* (1988) established that such a transition does indeed occur; moreover, it was shown that, in contrast to that of the ordinary percolation model, this transition is discontinuous (i.e. first order). In this letter, we prove the conjecture, also proposed by Mandelbrot (1983), that as $N \to \infty$, the thresholds $p_c(N)$ converge to the percolation threshold of the ordinary square site lattice model.

Below, we will first define the model precisely, and recapitulate the relevant results of Chayes *et al* (1988). We will then state and prove our theorem.

Consider the unit square $[0, 1]^2$, which we will denote by A_0. Let $N \geq 2$ be an integer and take $p \in (0, 1)$. The square A_0 is subdivided into N^2 (closed) smaller units

$$S_{i,j} = \left[\frac{i-1}{N}, \frac{i}{N}\right] \times \left[\frac{j-1}{N}, \frac{j}{N}\right] \tag{1}$$

$1 \leq i, j \leq N$, each of which is independently 'retained' with probability p or 'discarded' with probability $1-p$. The random subset of A_0 which has survived will be denoted by A_1. For the second iteration, each surviving square of A_1 is subjected to the same subdivision and retention problem, which generates the random set A_2. Further iterations follow the same scheme. Ultimately, one studies the statistical behaviour of the limiting set

$$A_\infty = \bigcap_j A_j. \tag{2}$$

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We will denote probabilities of events in the limiting set by $\text{Prob}_{N,p}[-]$. The relevant *percolation probability* in this model is

$$\theta(N,p) = \text{Prob}_{N,p}[A_\infty \text{ contains a left-to-right crossing of } [0,1]^2]. \quad (3)$$

The following basic properties—discussed by Mandelbrot (1983)—were established formally by Chayes *et al* (1988).

(I) Let $p_\varnothing(N) \equiv 1/N^2$. If $p \leq p_\varnothing(N)$, then with probability one, $A_\infty = \varnothing$, while if $p > p_\varnothing(N)$, then with positive (and computable) probability, $A_\infty \neq \varnothing$.

(II) For $p > p_\varnothing(N)$, the Hausdorff dimension of A_∞ (given that $A_\infty \neq \varnothing$) is, with probability 1, equal to $2 - |\log p/N|$.

(III) There is a $p_d(N)$, with $1/\sqrt{N} < p_d(N) \leq 1$, such that for $p < p_d(N)$, with probability 1, A_∞ is totally disconnected (or 'dust-like').

On a slightly more refined level, the following was proved by Chayes *et al* (1988).

(IV) For all $N \geq 2$, there is a $p_c(N) < 1$, such that for $p > p_c(N)$, there is *percolation* in the sense that $\theta(N,p) > 0$. Furthermore:

(A) $p_d(N) = p_c(N)$ and

(B) according to a number of criteria, the phase transition is discontinuous: e.g. $\theta(N, p_c(N)) > 0$.

Combining the above results, we see that the model has (at least) three phases: a null phase, a dust-like phase and a percolating phase; furthermore, the final transition, which occurs at $p_c(N)$, is discontinuous.

Concrete, rigorous estimates on the $p_c(N)$ are, as usual, difficult to achieve. For example, it was only established by Chayes *et al* (1988) that $p_c(3) < 1 - 10^{-4}$, to be contrasted with Mandelbrot's proposal (1983) that $p_c(3) \simeq \frac{8}{9}$. Nevertheless, it is easy to show that unless p exceeds the percolation threshold for the square site lattice, which we denote by $p_c(\text{site})$, there cannot be percolation in the fractal model for any value of N. (A formal proof of this will be supplied below.) Conversely, when $p > p_c(\text{site})$, then for large N, the system looks quite percolative—at least for the first few iterations†. Such considerations led Mandelbrot to conjecture that the large-N limit of $p_c(N)$ is, in fact, $p_c(\text{site})$. A proof of this conjecture is the subject of this letter.

Theorem. Let $p_c(N)$ denote the threshold value for percolation in the fractal percolation model with subdivision index N:

$$p_c(N) = \inf\{p \mid \theta(N,p) > 0\}$$

and let $p_c(\text{site})$ denote the percolation threshold for the usual square site lattice model. Then

$$\lim_{N \to \infty} p_c(N) = p_c(\text{site}).$$

Remark. Naively, one might expect that the existence of the above limit (though not its value) could have been established simply by monotonicity of the sequence $(p_c(N))$. Although—on the basis of the 'more is better' principle—we expect such monotonicity, this has not been rigorously demonstrated. It is, however, easy to show monotonicity of various subsequences; for example, $\theta(N,p) \leq \theta(N^2,p)$ which implies $p_c(N^2) \leq p_c(N)$.

† In particular, it was known to Chayes *et al* (1988) that if δ and m are fixed, for any $p > p_c(\text{site})$, one can produce an N large enough so that with probability exceeding $1 - \delta$, one will observe a left-to-right crossing after m iterations.

Proof. We first suppose that $p < p_c(\text{site})$. Denote by $\Gamma_N^{(n)}$ those configurations in the Nth model that exhibit a left-to-right crossing of $[0, 1]^2$ after n iterations of the process. Obviously, $\Gamma_N^{(n+1)} \subset \Gamma_N^{(n)}$ and $\theta(N, p) = \lim_{n\to\infty} \text{Prob}_{N,p}[\Gamma_N^{(n)}]$.

Now consider ordinary percolation on the square site (or any other 'regular' two-dimensional) lattice. It is known that, at the critical point, there is a universal (i.e. lattice-independent) constant $c < 1$, such that the probability of observing a left-to-right crossing of an $N \times N$ square is bounded above by c. This is, in fact, the key idea implicit in the proofs of Kesten (1980) and Russo (1981), and is discussed at some length by Chayes and Chayes (1986). For descriptions of the lattices for which such results can be proved, see Kesten (1982).

It is clear that the observation of such a crossing in the square site model is equivalent to the event $\Gamma_N^{(1)}$. Using the fact that the *best* possibility for the first n iterations is total retention (a minor miracle), it is straightforward to show that

$$\text{Prob}_{N,p}[\Gamma_N^{(n+1)} | \Gamma_N^{(n)}] \leq c. \tag{4}$$

Indeed, (4) is established by applying the Harris–FKG inequality (Harris 1960, Fortuin *et al* 1971) to each Bernoulli configuration $\omega \in \Gamma_N^{(n)}$. Thus, we have

$$\text{Prob}_{N,p}[\Gamma_N^{(n)}] \leq c^n \tag{5}$$

which tends to zero with n. Using the obvious monotonicity of $\theta(N, p)$ and the result (IVB), this establishes that for each N, $p_c(N) > p_c(\text{site})$.

We now probe the less trivial half of the theorem: namely, if $p > p_c(\text{site})$, then for all N sufficiently large, $p > p_c(N)$. However, it will pay if we first pause to again discuss the ordinary site model. In particular, we will outline a 'renormalised bond construction' along the lines of those developed by Aizenman *et al* (1983) and discussed in detail by Chayes and Chayes (1986). Denote by R_m the $3m \times m$ block: $R_m \equiv \{x | 0 \leq x_1 \leq 3m, 0 \leq x_2 \leq m\}$. Consider the event, depicted in figure 1, that there is left-to-right connected crossing and two top-to-bottom connected crossings, one lying in the left third and the other in the right third, of R_m, all of which are composed of retained (or 'occupied) sites. We denote this event by $\mathcal{B}_1 \equiv \mathcal{B}_1(m)$. For $p > p_c(\text{site})$, the probability of observing $\mathcal{B}_1$ tends to unity exponentially fast in m. Hence, for a fixed small ε_1 (to be chosen later), let us assume that m has been chosen so large that

$$\text{Prob}_p[\mathcal{B}_1] \geq 1 - \varepsilon_1. \tag{6}$$

(Here, of course, $\text{Prob}_p[-]$ denotes probability with respect to ordinary Bernoulli measure at density p.)

Now for an $N \times N$ square, with $N \gg m$, we may construct a block-bond lattice, denoted by $\mathbb{D}_N^{(m)}$, whose 'vertices' are $m \times m$ squares and whose 'edges' are translates and

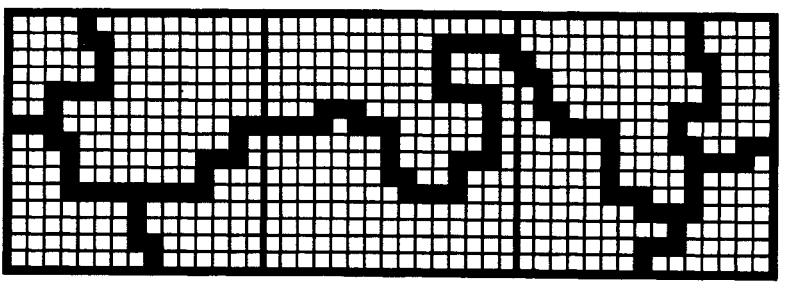

Figure 1. A block-bond event.

rotations of R_m (see figure 2). We may then do a percolation problem on $\mathbb{D}_N^{(m)}$ by declaring a $B \in \mathbb{D}_N^{(m)}$ to be 'occupied' when the corresponding translation and/or rotation of the event $\mathscr{B}_1$ occurs. Observe that these block-bond events have been constructed so that a connected chain of such bonds necessarily implies the existence of a true underlying connection. Although the block events are not independent, as would be the case in a typical percolation problem, each event is correlated with only a few neighbouring events. Furthermore (and most importantly), the system is at exceedingly high density.

Let us estimate the probability of achieving a left-to-right crossing of the $N \times N$ square via the block-bonds. In the absence of such a crossing, there would be a top-to-bottom crossing of the square consisting of a connected path of 'failed bonds,' as represented by events on the dual lattice $(\mathbb{D}_N^{(m)})^*$. A string of such failures of length n necessarily involves at least $n/4$ statistically independent failure events, since we may partition $\mathbb{D}_N^{(m)}$ into four disjoint sublattices within which the block-bond events are mutually independent. (See e.g. Russo (1978) for details of these types of arguments.) Let us denote by $\Xi_N^{(1)} \equiv \Xi_N^{(1)}(m)$ the event

$$\Xi_N^{(1)} = \{\omega | \text{ there is a left-to-right connected crossing of the } N \times N \text{ square consisting of block-bond events on } \mathbb{D}_N^{(m)}\}. \tag{7}$$

Since the minimum block-bond path across an $N \times N$ square necessarily consists of at least $N/2m - 1$ (overlapping) bond events, we may use the usual counting arguments to estimate

$$\text{Prob}_p[\Xi_N^{(1)}] \geqslant 1 - \frac{\text{constant}}{\varepsilon_1[1 - g(\varepsilon_1)^{1/4}]} \frac{N}{m} [g(\varepsilon_1)^{1/4}]^{N/2m} \tag{8}$$

for ε_1 sufficiently small. Here $g < 3$ is the relevant connectivity constant.

Thus far, we have only demonstrated that percolation implies percolation—and, in fact, we have used somewhat outdated methods. Let us now resume the topic of the *fractal* models when $p > p_c$(site). Suppose again that m is large and N still larger. If we focus, for the benefit of our first iteration, on the analogue of the event $\Xi_N^{(1)}$ (which obviously implies $\Gamma_N^{(1)}$) then, so far, we are in good shape. In order to withstand a second iteration, we will define a second-stage bond event. Consider a configuration in which (the analogue of) the event $\mathscr{B}_1$ occurs. In the rectangle $R_{m/N}$, those squares which produced the crossings—in addition to any other survivors—must subdivide. On these living regions, we again construct a block-bond lattice, i.e. a 'microblock'

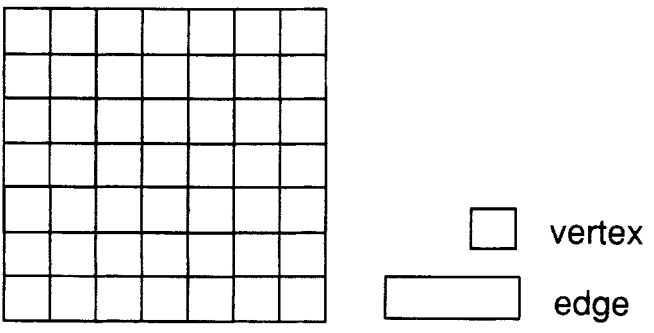

Figure 2. A block-bond lattice.

lattice, consisting of $3m \times m$ block units (the physical sizes of which are $3m/N^2 \times m/N^2$). Now we can watch for microblock-bond events which are small versions of (but entirely equivalent to) $\mathscr{B}_1$. Let us insist that after the second retention problem, the relevant paths that produced the event $\mathscr{B}_1$ are underscored by corresponding paths comprised of the bond events on the microblock lattice. We denote this event by $\mathscr{B}_2$. Explicitly, if the event $\mathscr{B}_1$ has occurred on the rectangle $R_{m/N}$, we consider the surviving region, $R_{m/N} \cap A_1$, and the microblock lattice, $\mathbb{D}(R_{m/N} \cap A_1)$, on $R_{m/N} \cap A_1$. The event $\mathscr{B}_2$ means that within $\mathbb{D}(R_{m/N} \cap A_1)$, there is a left-to-right crossing of $R_{m/N}$ by a connected path of microblock-bond events, in addition to top-to-bottom crossings of $R_{m/N}$ by such bonds in the left and right third.

Given the event $\mathscr{B}_1$, it is easy to estimate the event $\mathscr{B}_2$: using worst-case-scenario counting arguments, we have

$$\begin{aligned} \mathrm{Prob}_{N,p}[\mathscr{B}_2|\mathscr{B}_1] &\geqslant 1 - \frac{\text{constant}}{\varepsilon_1[1-g(\varepsilon_1)^{1/4}]} N^2[g(\varepsilon_1)^{1/4}]^{N/2m} \\ &\equiv 1 - f_{N,m}(\varepsilon_1). \end{aligned} \tag{9}$$

This yields the direct estimate

$$1 - \varepsilon_2 \equiv \mathrm{Prob}_{N,p}[\mathscr{B}_2] \geqslant (1 \quad _1)(1 - f_{N,m}(\varepsilon_1)) \geqslant 1 - \varepsilon_1 - f_{N,m}(\varepsilon_1). \tag{10}$$

The event $\Xi_N^{(2)} \subset \Gamma_N^{(2)}$ that $[0, 1]^2$ is crossed by second-stage bonds can now be estimated via equation (8) with ε_2 replacing ε_1.

We continue this procedure. At the $(k+1)$th stage, we define the event $\mathscr{B}_{k+1}$ by first requiring that the event $\mathscr{B}_1$ occurs, and then insisting that the relevant paths are 'underscored' by bond events comprised of translations and rotations of the event $\mathscr{B}_k$. We thus obtain the iterative inequality

$$1 - \varepsilon_{k+1} \geqslant 1 - \varepsilon_1 - f_{N,m}(\varepsilon_k) \tag{11a}$$

i.e.

$$\varepsilon_{k+1} \leqslant \varepsilon_1 + f_{N,m}(\varepsilon_k). \tag{11b}$$

The situation will obviously get out of control for a poor set of initial conditions. Conversely, it is seen that for a reasonable set of initial conditions, the (ε_k) will hover just above ε_1. All such considerations amount to an analysis of the fixed-point equation

$$\varepsilon^* = \varepsilon_1 + f_{N,m}(\varepsilon^*). \tag{12}$$

It is easy to see that, provided $g(\varepsilon_1)^{1/4} < 1$, for N sufficiently large, there is a non-trivial fixed point $\varepsilon^*(N, m, \varepsilon_1)$ with $\varepsilon^* \to \varepsilon_1$ as $N \to \infty$. The resulting ε^* may then be fed into equation (8), which provides a lower estimate on $\theta(N, p)$. Hence, all we need do is choose a large m, driving ε_1 down, and then ensure that N is selected large enough. For such N, we have $p > p_c(N)$, the desired inequality. □

Concluding Remarks. It is obvious that these techniques also provide a more efficient proof of the original question of the non-triviality of $p_c(N)$. It is, at this point, possible to obtain bad (as opposed to ridiculous) bounds for small N—e.g. $N \geqslant 6$. In the large-N regime, our proof demonstrates that if N is a huge (but fixed) multiple of the correlation length of the density-p site model, then $p > p_c(N)$.

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